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A Paradox in Optimal Flow Control of M/M/m Queues

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***Abstract*— Optimal flow control problems of multiple-server (M/M/m) queueing systems are studied. Due to enhanced flexibility of the decision making, intuitively, we expect that grouping together separated systems into one system provides improved performance over the previously separated systems. This paper presents a counter-intuitive result. We consider a noncooperative optimal flow control problem of M/M/m queueing systems where each player strives to optimize unilaterally its own power where the power of a player is the quotient of the throughput divided by the mean response time for the player. We report a counter-intuitive case where the power of every user degrades after grouping together $K(>1)$ separated M/M/N systems into a single M/M/($K\times N$) system. Some numerical results are presented.**

I. INTRODUCTION

In computer and communication networks, flow control is one of the important means to utilize limited system resources effectively and to guarantee proper Quality of Service (QoS). Flow control adjusts the input flow (throughput) in order to provide the best performance. Considering an optimal flow control problem, one may face the trade-off between the throughput and the response time. These two performance measures are mutually contradictory, that is, if one improves the system throughput, then the system response time degrades, and vice versa. Therefore, as the utility of each user (player), we consider the power that is the quotient of the throughput divided by the average response time for the user (see, e.g., Giessler et al., [8], Kleinrock [16] [17]).

We consider a system where multiple users (players) share an M/M/m queueing system and where the utility (or the performance) of a player is the power. We can think of two typical performance objectives. One is *overall optimization* and the other is *noncooperative optimization*. In overall optimization, a single performance measure that is the total sum of the powers of all players is optimized. In noncooperative optimization, each player strives to optimize unilaterally its own power given the decisions of others. It is regarded as a noncooperative game, and the equilibrium is called a *Nash equilibrium*. Throughout this paper, we concentrate on the *Nash equilibrium* [23].

Douligeris and Mazumdar [7], and Zhang and Douligeris [25] studied algorithms to obtain Nash equilibria in flow control of M/M/1 queueing systems with multiple users. The performance objective was the maximization of the power. They proposed 'greedy' algorithms, and showed convergence properties of their algorithms. State dependent flow control was analyzed by Hsiao and Lazar [12], and Korilis and Lazar [18]. They considered a closed queueing network model, and maximized the average throughput subject to an upper bound on the average response time. In particular, Korilis and Lazar [18] derived the existence of equilibria using fixed-point theorems. Altman et al. [1] combined flow control and routing in a network model with several parallel links. Lazar [21] studied optimal flow control problems of an M/M/m queue where one player maximizes the throughput subject to the constraint that the average time delay should not exceed a specified value. In this paper, we deal with flow control problems of multiple-server (M/M/m) queueing systems that have multiple players, where each player strives to optimize unilaterally its power.

Consider the operation of grouping together separated systems into one system. Due to enhanced flexibility in resource utilization, we expect that grouping system together improves the system performance. For example, it has been shown analytically that grouping together separated systems provides an improved average response time over those of previously separated systems (see [15]). Therefore, we expect performance improvement in terms of power by grouping together separate systems in optimal flow control.

In noncooperative optimization of routing in networks and load balancing in distributed systems, however, it is known that the following phenomenon can occur: Grouping separated systems together and/or adding the system capacity may sometimes degrade of the utilities for all users. The first example is the Braess paradox [3]. Other examples have been presented, for example, by Bean et al. [2], Calvert et al. [4], Cohen and Jeffries [5], Cohen and Kelly [6], Kameda et al. [13], Kameda and Pourtallier [14], Korilis et al. [19] [20], Roughgarden and Tardos [24] and so on. However, most of the previous work are on routing (or load balancing) problems, and the paradox is seen in very limited models. Flow control is essentially different from them. In optimal flow control, it seems that no such paradoxes have been reported yet.

In this paper, we show that a paradox similar to the above ones may occur in noncooperative optimal flow control of M/M/m queues. That is, we show a case where unification leads to the degradation of the power of all players in noncooperative optimal flow control. In the following, we first formulate both overall and noncooperative flow control problems to maximize the power as nonlinear programming problems. Then, we show a case where a paradox occurs in noncooperative flow control of M/M/m queueing systems.

The rest of this paper is organized as follows. In Section II, we describe a queueing system model and formulate

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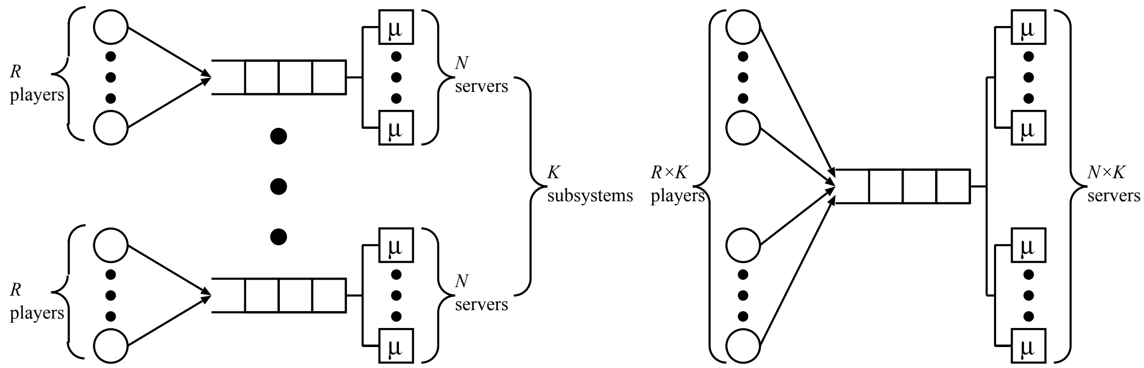

Fig. 1. **[Left]** A group of K M/M/N queues (S_I) vs. **[Right]** An M/M/($N\times K$) queue (S_U).

overall and noncooperative optimal flow control problems. In Section III, we show the case of a paradox in noncooperative flow control. Finally, in Section IV, we conclude this paper. In the Appendix, we show the existence and the uniqueness of solutions to the optimization problems.

II. THE MODEL AND PROBLEM FORMULATION

We consider two queueing system models as shown in Fig. 1. One system, called S_I, consists of K subsystems each of which consists of a separated M/M/N queue (N exponential servers with Poisson arrivals of jobs) and R independent players. The processing time of a job at each server is independent and identically and exponentially distributed with mean $1/\mu$. The other system, called S_U, results from grouping together all the above separated M/M/N queues. That is, the system consists of one M/M/$(K\times N)$ queue and $K\times R$ independent players.

Each subsystem of S_I and the system S_U have m exponential servers with a single loss-less queueing line (M/M/m) and r distinguishable players where m corresponds N and $K\times N$, and r corresponds R and $K\times R$, respectively. Each player sends the multiple-server an arrival stream of jobs that are mutually independent and form a Poisson arrival process at the rate of λ_l jobs per unit time, $l=1,2,\ldots,r$. For convenience, we define following symbols: the total arrival flow $\Lambda=\sum_{l=1}^{r}\lambda_l$ and the vector $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_r)$. Furthermore, assuming that jobs in the system are processed according to first-come-first-served (FCFS) discipline, we model the system as an M/M/m queueing system with total arrival rate Λ, and m servers at processing rate μ. Then, the average response time of player l is given by

$$T_l(\boldsymbol{\lambda})=T_l(\Lambda)=\begin{cases}\dfrac{B_m(\Lambda)}{m\mu-\Lambda}+\dfrac{1}{\mu}, & \text{if } 0\le\Lambda<m\mu,\\ \infty, & \text{if } \Lambda\ge m\mu,\end{cases}\tag{1}$$

for $l=1,2,\ldots,r$, where

$$B_m(\Lambda)=\left[1+\sum_{k=0}^{m-1}\frac{m!(1-\frac{\Lambda}{m\mu})}{k!(\frac{\Lambda}{\mu})^{m-k}}\right]^{-1}\tag{2}$$

is the probability that all servers are busy (called *the Erlang delay formula*). Since the processing time of an arbitrary job is identically distributed, the average response time of an arbitrary job is also identical. Denote by T the overall average response time. Then, $T(\Lambda)=T_l(\Lambda)$ for $l=1,2,\ldots,r$.

As the utility for each user in the flow control problems, we consider the power for the user. Since the throughput is equivalent to the rate of the arrival flow (i.e., the arrival rate), from [17], the overall power P of the system and the power P_l of player l are given as:

$$P=P(\Lambda)=\begin{cases}\dfrac{\Lambda}{T(\Lambda)}, & \text{if } 0\le\Lambda\le m\mu,\\ 0, & \text{if } \Lambda>m\mu,\end{cases}\tag{3}$$

and

$$P_l=P_l(\boldsymbol{\lambda})=\begin{cases}\dfrac{\lambda_l}{T(\Lambda)}, & \text{if } 0\le\lambda_l\le m\mu-\sum_{j\ne l}\lambda_j,\\ 0, & \text{otherwise},\end{cases}\tag{4}$$

$l=1,2,\ldots,r$, respectively. By noting that $\Lambda=\sum_l\lambda_l$, we have $P=\sum_l P_l$. P and P_l are non-negative, and zero for $m\mu\le\Lambda$.

We formulate for the system described above two typical optimal flow control problems: the overall optimization problem and the noncooperative optimization problem. The overall optimization maximizes the overall power of the system. On the other hand, under noncooperative optimization, each player strives to maximize unilaterally its own power. The overall and noncooperative optimization problems in an M/M/m queueing system with m players are presented as follows:

(I) The overall optimization: A single agent maximizes the overall power of the system, that is, it strives to find $\widetilde{\Lambda}$ that satisfies

$$\widetilde{P}=\max_{\Lambda\ge0}P(\Lambda).$$

Then, the agent distributes equally to each player l the power $\widetilde{P}_l$, and thus the throughput $\widetilde{\lambda}_l$. Thus, $\widetilde{P}_l=\widetilde{P}/r$ and $\widetilde{P}_l=\widetilde{P}/r$.

(II) The noncooperative optimization: Each player strives to maximize unilaterally its power, that is, the noncooperative optimization is to find $\widehat{\lambda}_l$ for each $l=$

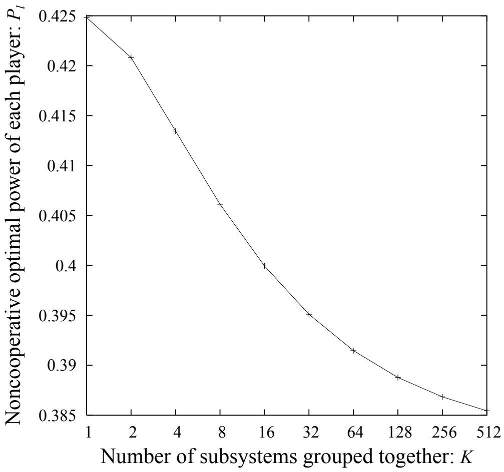

Fig. 2. [**Noncooperative optimization**]: The graph shows a counter-intuitive trend of noncooperative optimization. That is, the power of each player $\widehat{P}_l$ *decreases* as the number K of subsystems grouped together *increases*, for $N = 11$ and $R = 11$.

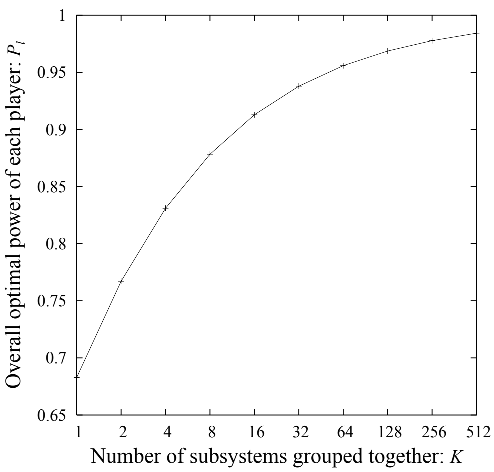

Fig. 3. [**Overall optimization**]: The graph shows the naturally-expected trend of overall optimization. That is, the power of each player $\widetilde{P}_l$ increases as the number K of subsystems grouped together increases, for $N = 11$ and $R = 11$.

$1, 2, \ldots, r$, that satisfies:

$$\widehat{P}_l = \max_{\lambda_l \geq 0} P_l(\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \lambda_l, \ldots, \widehat{\lambda}_r).$$

We can obtain overall and noncooperative optimal solutions by solving simple nonlinear equations (see Appendix).

III. THE CASE OF A PARADOX

In this section, we show a case where a paradox occurs. We note that $\widetilde{P}_l$ is the overall optimal power for player l. Without loss of generality, we assume a time scale such that $\mu = 1$.

We compare the two systems presented in Section II (Fig. 1): S_I – a group of K M/M/N queues (left) and S_U – an M/M/($N \times K$) queue (right). In the former, the flow control problems are concerned with each separated M/M/N queue. On the other hand, in the latter, the flow control problems are concerned with the M/M/($K \times N$) system. We note that the latter is the result of grouping the queues of the former.

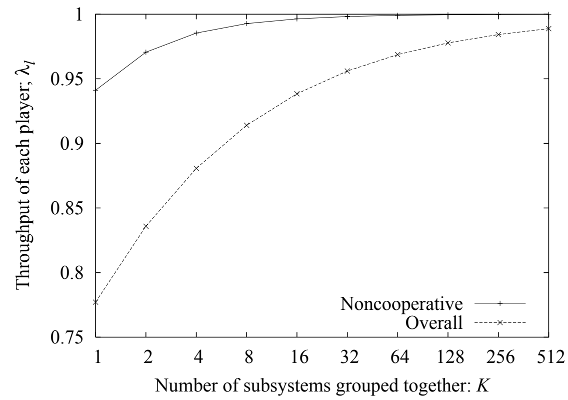

Fig. 4. [**Overall and noncooperative optimization**]: The graph shows how the throughputs of each player $\widehat{\lambda}_l$ (noncooperative) and $\widetilde{\lambda}_l$ (overall) change as the number K of subsystems grouped together increases, for $N = 11$ and $R = 11$.

Table I and Fig. 2 illustrate how the noncooperative optimal power of each player $\widehat{P}_l$ depends on the number K of subsystems grouped together, for the case when the number N of servers and R of players in each subsystem is 11 respectively. The degree of paradox δ in Table I is defined as follows: Denote the noncooperative optimal power of each player in S_I and S_U by $\widehat{P}_l$ and $\widehat{P}'_l$ respectively. Then

$$\delta = \frac{\widehat{P}_l - \widehat{P}'_l}{\widehat{P}_l} \tag{5}$$

is the degree of paradox with K subsystems, where we note that $\widehat{P}_l$ is equivalent to $\widehat{P}'_l$ when $K = 1$. Obviously, a paradox occurs when $\delta > 0$.

It looks counter-intuitive that in Fig. 2, the power of each user $\widehat{P}_l$ decreases as the number K of the systems grouped together increases. This means that we should not always expect that system unification improves the system performance in the noncooperative flow control even though we feel intuitively that unification must increase the performance. As seen in Table I, the degree of paradox reaches about 10% in the case where $N = 512$, and goes further up as K further increases.

On the other hand, Fig. 3 shows that the overall optimal power of each player $\widetilde{P}_l$ increases as the number K of subsystems grouped together increases, for the case of both the number N of servers and R of players in each subsystem equal 11. This agrees to our natural intuition.

Fig. 4 shows the throughputs of each player given in noncooperative and overall flow control where $N = R = 11$. From Fig. 4, we observe that the throughput given by noncooperative flow control is larger than the throughput given by overall flow control for any K. z

Remark 1: We have investigated the paradoxical behaviors exhaustively regarding the various combinations of the numbers of servers, and players and the number of

TABLE I

NONCOOPERATIVE OPTIMAL POWER OF EACH PLAYER, $N = 11$ AND $R = 11$.

K	1	2	4	8	16	32	64	128	256	512
power	0.4248	0.4208	0.4135	0.4061	0.3999	0.3951	0.3915	0.3888	0.3868	0.3854
degree of paradox	0.000	0.009	0.027	0.044	0.059	0.070	0.078	0.085	0.089	0.093

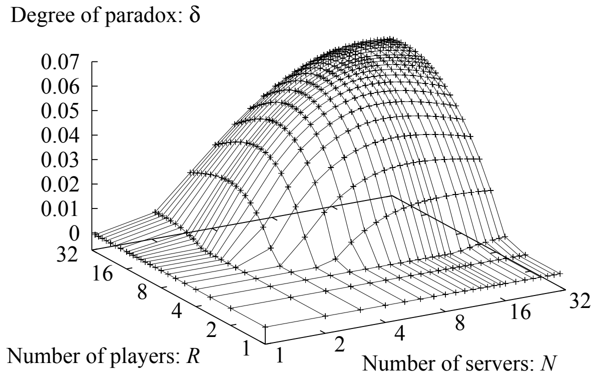

Fig. 5. [**Noncooperative optimization**]: The graph shows the degree of paradox δ in noncooperative flow control for the various combinations of N and R in the case where $K = 16$, where we set the value of δ to be zero in the cases where $\delta < 0$.

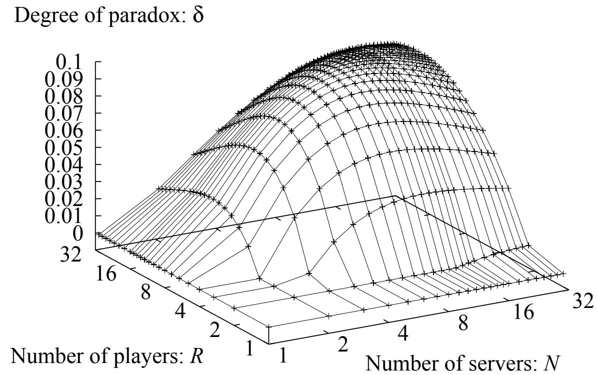

Fig. 6. [**Noncooperative optimization**]: The graph shows the degree of paradox δ in noncooperative flow control for the various combinations of N and R in the case where $K = 512$, where we set the value of δ to be zero in the cases where $\delta < 0$.

subsystems to be grouped together. Figs. 5 and 6 show the degree of paradox for the various combinations of the values of N and R in the two cases where $K = 16$ and 512, respectively, where we set the value of δ to be zero in the cases where $\delta < 0$. Comparing both figures, we see that the combinations of N and R for which the paradox occurs increases when K is larger. Also, we observe that the degree of paradox tends to be larger when the values of N is nearly equal to R.

Thus we see that the worst case of paradox occurs in the case when $N = 11$ and $R = 11$ when K is a large value (see the example presented earlier in this section). To give some idea on the results of overall results, we present in Fig. 7 the case where $N = 1$. In overall optimization, each player always enjoys the increase of the power as the number K of subsystems grouped together increases. On the other hand, in noncooperative optimization, for the values of K from 1 till about 10, each player benefits from the increase of the power although the size of the increase is smaller than in the case of overall optimization. For the values of K larger than 11, the power of each player decreases as K increases, which shows the same trend as the above-mentioned paradoxical result.

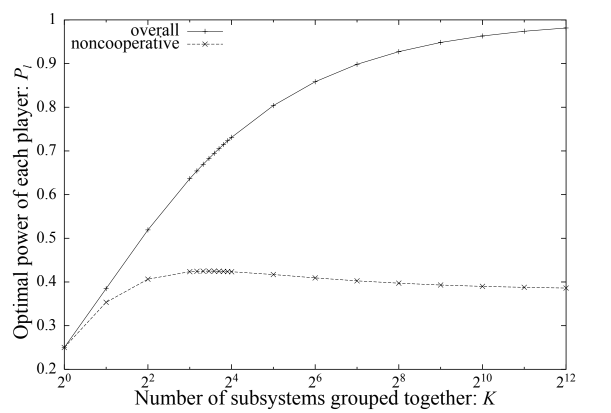

Fig. 7. [**Overall and noncooperative optimization**]: The graph shows how the powers of each player $\hat{P}_l$ (noncooperative) and $\widetilde{P}_l$ (overall) change as the number K of subsystems grouped together increases, for $N = 1$ and $R = 1$.

IV. CONCLUSION

In this paper, we have studied the existence of a paradox in noncooperative flow control in M/M/m queues. We have formulated the overall and noncooperative optimal flow control problems of maximizing the power and have shown the existence and uniqueness of their solutions. We have found a paradoxical behavior in noncooperative optimal flow control similar to the Braess paradox of noncooperative optimal routing.

We have interest in this research because optimal flow control may essentially be different from optimal routing and also because to our knowledge paradoxes have only been found to occur in limited types of problems not including optimal flow control. Our example of paradox found in optimal flow control would suggest that paradoxes could also occur in various other types of problems.

APPENDIX: EXISTENCE AND UNIQUENESS OF SOLUTIONS

First, we show that there exist solutions to problems (I) and (II). For M/M/1 queueing system models, there have been obtained the closed form solutions to overall and noncooperative optimization (see e.g., [7]). It seems, however, that they have not been shown for M/M/m models for $m > 1$. Furthermore, in this Appendix, the uniqueness of the solutions is proved.

Lemma 1: 1) For $0 \le \Lambda \le m\mu$, the overall power, $P(\Lambda)$ is strictly concave in Λ.

2) For $0 \le \lambda_l \le m\mu - \sum_{j \ne l} \lambda_j$, the power of player l, $P_l(\boldsymbol{\lambda})$, is strictly concave in λ_l for $l = 1, 2, \ldots, r$.

Proof: Since 1) is equivalent to 2) in the case where $r = 1$, we prove 2). The second derivative of $P_l(\boldsymbol{\lambda})$ with respect to λ_l is as follows:

$$\begin{aligned} \frac{\partial^2 P_l(\boldsymbol{\lambda})}{\partial \lambda_l^2} &= 2\frac{\partial}{\partial \lambda_l}\frac{1}{T(\Lambda)} + \lambda_l \frac{\partial^2}{\partial \lambda_l^2}\frac{1}{T(\Lambda)} \\ &= -2\frac{1}{T(\Lambda)^2}\frac{\partial T(\Lambda)}{\partial \lambda_l} + \lambda_l \frac{\partial^2}{\partial \lambda_l^2}\frac{1}{T(\Lambda)}. \end{aligned} \quad (6)$$

Now, we wish to show (6) is negative. From [9], [10] and [22], the response time $T(\Lambda)$ is strictly increasing in ρ, where $\rho = \sum \lambda_l/(m\mu) = \Lambda/(m\mu)$, and hence $T(\Lambda)$ is also strictly increasing in λ_l. Therefore, the first term of (6) is negative.

We rewrite the second term of (6) as follows:

$$\lambda_l \frac{\partial^2}{\partial \lambda_l^2}\frac{1}{T(\Lambda)} = \lambda_l \frac{\partial^2}{\partial \rho^2}\frac{1}{T(\Lambda)}\left(\frac{\partial \rho}{\partial \lambda_l}\right)^2.$$

From [11], we have $d^2T(\Lambda)^{-1}/d\rho^2 < 0$. Note that $\partial\rho/\partial\lambda_l = 1/(m\mu)$. Therefore, the second term of (6) is nonpositive. From these results, it is shown that (6) is negative. ■

We define the following equation:

$$g(\Lambda) = \frac{T(\Lambda)}{T'(\Lambda)} - \frac{\Lambda}{r}, \quad (7)$$

where $T'(\Lambda) = dT(\Lambda)/d\Lambda$.

Lemma 2: Any solution to problem (II) is symmetric (i.e., $\widehat{\lambda}_l = \widehat{\lambda}_{l'} = \widehat{\Lambda}/r,\ l, l' = 1, 2, \ldots, r$), where $g(\widehat{\Lambda}) = 0$.

Proof: We denote a solution to (II) by $\widehat{\boldsymbol{\lambda}}$. Note that $\widehat{\boldsymbol{\lambda}}$ satisfies $\widehat{\lambda}_l > 0$ for all $l = 1, 2, \ldots, r$ and $\sum_l \widehat{\lambda}_l < m\mu$. Then we see that $\widehat{\boldsymbol{\lambda}}$ is a solution to $\partial P_l(\boldsymbol{\lambda})/\partial\lambda_l = 0,\ l = 1, 2, \ldots, r$, which is equivalent to

$$\lambda_l \frac{\partial T(\Lambda)}{\partial \lambda_l} - T(\Lambda) = 0,\ l = 1, 2, \ldots, r. \quad (8)$$

Since $T(\Lambda)$ depends only on Λ, we see $\partial T(\Lambda)/\partial\lambda_l = \partial T(\Lambda)/\partial\lambda_{l'} = dT(\Lambda)/d\Lambda,\ l, l' = 1, 2, \ldots, r$. Then, any solution to problem (II) is symmetric, and given by $g(\widehat{\Lambda}) = 0$ and $\widehat{\lambda}_l = \widehat{\Lambda}/r,\ l = 1, 2, \ldots, r$. ■

Therefore, we obtain the following propositions:

Proposition 1: 1) The solution to overall optimization problem (I) is unique, and it is a solution to the following equation:

$$\frac{T(\Lambda)}{T'(\Lambda)} - \Lambda = 0. \quad (9)$$

2) The solution to noncooperative optimization problem (II) is unique, and it is a solution to the following equation:

$$g(\Lambda) = \frac{T(\Lambda)}{T'(\Lambda)} - \frac{\Lambda}{r} = 0. \quad (10)$$

Proof: 1) We denote a solution to (I) by $\widetilde{\Lambda}$. Note that P is positive for $0 < \Lambda < m\mu$, and zero for $\Lambda \le 0$ and $\Lambda \ge m\mu$. Also from Lemma 1, P is strictly concave for $0 \le \Lambda \le m\mu$. Then problem (I) has a unique solution, and $\widetilde{\Lambda}$ is a solution to $dP(\Lambda)/d\Lambda = 0$ for $0 < \Lambda < m\mu$, which is equivalent to (9).

2) From Lemma 2, it is obvious that a solution to problem (II) is given by $g(\Lambda) = 0$, which is equivalent to (10). If $g(\Lambda)$ is a strictly decreasing function of Λ for $0 \le \Lambda \le m\mu$, and satisfies $g(0) > 0$ and $g(m\mu) < 0$, there exists a unique solution to (10). The derivative of g with respect to Λ is

$$\begin{aligned} \frac{dg(\Lambda)}{d\Lambda} &= \frac{T'(\Lambda)^2 - T(\Lambda)T''(\Lambda)}{T'(\Lambda)^2} - \frac{1}{r} \\ &= -\frac{T(\Lambda)T''(\Lambda) - 2T'(\Lambda)^2}{T'(\Lambda)^2} - 1 - \frac{1}{r}, \end{aligned}$$

where $T''(\Lambda) = d^2T(\Lambda)/d\Lambda^2$. In [11], it is shown that $d^2T^{-1}/d\Lambda^2 < 0$, that is to say that $T(\Lambda)T''(\Lambda) - 2T'(\Lambda)^2 > 0$. Then, we have $dg(\Lambda)/d\Lambda < 0$ for $0 \le \Lambda \le m\mu$. Therefore, $g(\Lambda)$ is a strictly decreasing function of Λ.

Next, we show $g(0) > 0$ and $g(m\mu) < 0$. From [9], [10], and [22], the function $T(\Lambda)$ is increasing and strictly convex in Λ for $0 \le \Lambda \le m\mu$. Then $T'(\Lambda)$ is positive for $\Lambda = 0$. From (1), $T(\Lambda)$ is also positive for $\Lambda = 0$. Therefore, from (10) we have $g(0) > 0$.

The derivatives of $T(\Lambda)$ and $B_m(\Lambda)$ are

$$\frac{dT(\Lambda)}{d\Lambda} = \frac{B'_m(\Lambda)(m\mu - \Lambda) + B_m(\Lambda)}{(m\mu - \Lambda)^2},$$

and

$$\begin{aligned} B'_m(\Lambda) &= \frac{dB_m(\Lambda)}{d\Lambda} \\ &= \frac{B_m(\Lambda)[m\mu\Lambda(1 - B_m(\Lambda)) + m(m\mu - \Lambda)^2]}{m\mu\Lambda(m\mu - \Lambda)}, \end{aligned}$$

respectively. Then if $\Lambda \ne 0$, $g(\Lambda)$ is rewritten as follows:

$$g(\Lambda) = \frac{m\Lambda(m\mu - \Lambda)(\mu B_m(\Lambda) + m\mu - \Lambda)}{B_m(\Lambda)[m\mu\Lambda + (1 - B_m(\Lambda)) + m(m\mu - \Lambda)^2]} - \frac{\Lambda}{r}.$$

Since $B_m(m\mu) = 1$, we obtain $g(m\mu) < 0$. Therefore, (10) has a unique solution. ■

Remark 2: Each of eqs. (9) and (10) has a single variable, respectively. Therefore, we can obtain the solutions simply by using an iterative algorithm (e.g., binary and golden section searches).

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